Keywords, chapter keywords and abstracts

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- Full t**itle:** Time-Variant and Quasi-separable Systems: Matrix Theory, Recursions and Computations

Book keywords: Dynamical system, time-variant, quasi-separability, semi-separability, SSS, linear algebra, state-space, system identification, state estimation, matrix algebra, interpolation, recursions, lossless embedding

Book abstract: This book presents the theory of time-variant systems, or equivalently, recursive matrix operations, based on a novel unifying and elementary computational approach. It consists of a set of 11 lectures on basics, followed by several deeper applications of systems theory to matrix algebra, illustrating the power of the basic theory. As a result, it offers an accessible, complete and theoretically fully motivated unifying workbench for electrical engineers, numerical analysts and signal-processing engineers, who need to understand how systems behave, how computational efficiency can be enhanced algorithmically and how recursive algorithms can be mastered. To achieve this, the approach is fully vested in the basic principles of modern system theory as they present themselves in time-variant environments. As a further illustration of the power of the basic theory, the book presents original solutions to several major issues in matrix algebra, to wit efficient Moore–Penrose inversion, LU and spectral factorizations, constrained approximation and interpolation, scattering theory and embedding theory.

Chapter 1:

- **Chapter title:** A First Example: Optimal Quadratic Tracking

Keywords: Tracking, quadratic optimization, Bellman, dynamic programming

Abstract: The book starts out with a motivating chapter to answer the question: Why is it worthwhile to develop system theory? To do so, we jump fearlessly in the very center of our

methods, using a simple and straight example in optimization: optimal tracking. Although optimization is not our leading subject – which is system theory – it provides for one of the main application areas, namely the optimization of the performance of a dynamical system in a time-variant environment (for example, driving a car or sending a rocket to the moon). The chapter presents a recursive matrix algebra approach to the optimization problem, known as dynamic programming. Optimal tracking is based on a powerful principle called "dynamic programming," which lies at the very basis of what "dynamical" means.

Chapter 2:

- **Chapter title:** Dynamical Systems

Keywords: dynamical system, Nerode, state, reachability, observability, behavior

Abstract: What is a *system*? What is a *dynamical system*? Systems are characterized by a few central notions: their *state* and their *behavior* foremost, and then some derived notions such as *reachability* and *observability*. These notions pop up in many fields, so it is important to understand them in nontechnical terms. This chapter therefore introduces what people call a narrative that aims at describing the central ideas. In the remainder of the book, the ideas presented here are made mathematically precise in concrete numerical situations. It turns out that a sharp understanding of just the notion of *state* suffices to develop most if not the whole mathematical machinery needed to solve the main engineering problems related to systems and their dynamics.

Chapter 3:

- **Chapter title:** LTV (Quasi-separable) Systems

Keywords: time-variant, LTV, state equations, block diagonals, nest algebra, shift-Z

Abstract: This chapter starts developing our central linear time-variant (LTV) prototype environment, a class that coincides perfectly with linear algebra and matrix algebra, making the correspondence between system and matrix computations a mutually productive reality. People familiar with the classical approach, in which the z-transform or other types of transforms are used, will easily recognize the notational or graphic resemblance, but there is a major difference:

everything stays in the context of elementary matrix algebra, no complex function calculus is involved, and only the simplest matrix operations, namely addition and multiplication of matrices, are needed. Appealing expressions for the state-space realization of a system appear, as well as the global representation of the input–output operator in terms of four block diagonal matrices $\{A, B, C, D\}$ and the (now noncommutative but elementary) causal shift Z. The consequences for and relation to linear time-invariant (LTI) systems and infinitely indexed systems are fully documented in *-sections, which can be skipped by students or readers more interested in numerical linear algebra than in LTI system control or estimation.

Chapter 4:

- **Chapter title:** System Identification

Keywords: identification, causal, anti-causal, realization, partial, Kronecker, Ho-Kalman

Abstract: From this point on, main issues in system theory are tackled. The very first, considered in this chapter, is the all-important question of system identification. This is perhaps the most basic question in system theory and related linear algebra, with a large pedigree starting from Kronecker's characterization of rational functions to its elegant solution for time-variant systems presented here. Identification, often also called *realization*, is the problem of deriving the internal system's equations (called state-space equations) from input–output data. In this chapter, we only consider the causal, or block-lower triangular case, although the theory applies just as well to an anti-causal system, for which one lets the time run backward, applying the same theory in a dual form.

Chapter 5:

- **Chapter title:** State Equivalence, State Reduction

Keywords: minimality, system equivalence, Hankel operator, matrix factorization, orthonormal bases

Abstract: In this chapter, we consider the central issue of minimality of the state-space system representation, as well as equivalences of representations. The question introduces important new basic operators and spaces related to the state-space description. In our time-variant context,

what we call the Hankel operator plays the central role, via a minimal composition (i.e., product), of a reachability operator and an observability operator. Corresponding results for LTI systems (a special case) follow readily from the LTV case. In a later starred section and for deeper insights, the theory is extended to infinitely indexed systems, but this entails some extra complications, which are not essential for the main, finite-dimensional treatment offered, and can be skipped by students only interested in finite-dimensional cases.

Chapter 6:

- **Chapter title:** Elementary Operations

Keywords: matrix operations, system addition, system multiplication, system inversion, inner system, outer system, lossless

Abstract: This chapter is on elementary matrix operations using a state-space or, equivalently, quasi-separable representation. It is a straightforward but unavoidable chapter. It shows how the recursive structure of the state-space representations is exploited to make matrix addition, multiplication and elementary inversion numerically efficient. The notions of outer operator and inner operator are introduced as basic types of matrices playing a central role in various specific matrix decompositions and factorizations to be treated in further chapters.

Chapter 7:

- **Chapter title:** Inner Operators and External Factorization

Keywords: coprime factorization, external representation, Bezout equations, recursive QR

Abstract: Several types of factorizations solve the main problems of system theory (e.g., identification, estimation, system inversion, system approximation, and optimal control). The factorization type depends on what kind of operator is factorized, and what form the factors should have. This and the following chapter are, therefore, devoted to the two main types of factorization: this chapter treats what is traditionally called coprime factorization, while the next is devoted to inner–outer factorization. Coprime factorization, here called “external factorization” for more generality, characterizes the system’s dynamics and plays a central role in system characterization and control issues. A remarkable result of our approach is the derivation

of Bezout equations for time-variant and quasi-separable systems, obtained without the use of Euclidean divisibility theory. From a numerical point of view, all these factorizations reduce to recursively applied QR or LQ factorizations, applied on appropriately chosen operators.

Chapter 8:

- **Chapter title:** Inner-Outer Factorization

Keywords: inner–outer factorization, QR factorization, LQ factorization, nonlinear, inner–outer, outer–inner, minimal phase

Abstract: This chapter considers likely the most important operation in system theory: inner–outer and its dual, outer–inner factorization. These factorizations play a different role than the previously treated external or coprime factorizations, in that they characterize properties of the inverse or pseudo-inverse of the system under consideration, rather than the system itself. Important is that such factorizations are computed on the state-space representation of the original, that is, the original data. Inner–outer (or outer–inner) factorization is nothing but recursive "QR factorization," as was already observed in our motivational Chapter 2, and outer–inner is recursive "LQ factorization," in the somewhat unorthodox terminology used in this book for consistency reasons: QR for "orthogonal Q with a right factor R' and LQ for a "left factor" L with orthogonal Q'. These types of factorizations play the central role in a variety of applications (e.g., optimal tracking, state estimation, system pseudo-inversion, and spectral factorization) to be treated in the following chapters. We conclude the chapter showing how the time-variant, linear results generalize to the nonlinear case.

Chapter 9:

- **Chapter title:** Application: The Kalman Filter

Keywords: Kalman-filtering, state-estimation, llse, stochastic-system, smoothing

Abstract: The set of basic topics then continues with a major application domain of our theory: linear least-squares estimation (llse) of the state of an evolving system (aka Kalman filtering), which turns out to be an immediate application of the outer–inner factorization theory developed

in Chapter 8. To complete this discussion, we also show how the theory extends naturally to cover the smoothing case (which is often considered "difficult").

Chapter 10:

- **Chapter title:** Polynomial Representations

Keywords: time-variant shift, polynomials, rational functions, nest algebra, Bezout equations, deadbeat, staircase matrices

Abstract: Two chapters conclude the basic course.

This chapter presents an alternative theory of external and coprime factorization, using polynomial denominators in the noncommutative time-variant shift Z rather than inner denominators as done in the chapter on inner–outer theory. "Polynomials in the shift Z" are equivalent to block-lower matrices with a support defined by a (block) staircase, and are essentially different from the classical matrix polynomials of module theory, although the net effect on system analysis is remarkably similar. The polynomial method differs substantially and in a complementary way from the inner method. It is computationally simpler but does not use orthogonal transformations. It offers the possibility of treating highly unstable systems using unilateral series. Also, this approach leads to famous Bezout equations that, as mentioned in the abstract of Chapter 7, can be derived without the benefit of Euclidean divisibility methods.

Chapter 11:

- **Chapter title:** Quasi-separable Moore–Penrose Inversion

Keywords: Moore–Penrose-inverse, quasi-stationary

Abstract: This chapter considers the Moore–Penrose inversion of full matrices with quasi-separable specifications, that is, matrices that decompose into the sum of a block-lower triangular and a block-upper triangular matrix, whereby each has a state-space realization given. We show that the Moore–Penrose inverse of such a system has, again, a quasi-separable specification of the same order of complexity as the original and show how this representation can be recursively computed with three intertwined recursions. The procedure is illustrated on a 4 × 4 (block) example.

Chapter 12:

- **Chapter title:** LU (Spectral) Factorization

Keywords: LU factorization, spectral factorization, Wiener filtering

Abstract: The following five chapters exhibit further contributions of the theory of time-variant and quasi-separable systems to matrix algebra. This chapter treats LU factorization, or, equivalently, spectral factorization, which is another, often occurring type of factorization of a quasi-separable system. This type of factorization does not necessarily exist and, when it exists, could traditionally not be computed in a stable numerical way (Gaussian elimination). Here we present necessary and sufficient existence conditions and a stable numerical algorithm to compute the factorization using orthogonal transformations applied to the quasi-separable representation.

Chapter 13:

- **Chapter title:** Matrix Schur Interpolation

Keywords: Schur parametrization, constrained interpolation, positive-definite matrices, maximum entropy, PR completion

Abstract: This chapter introduces a different kind of problem, namely direct constrained matrix approximation via interpolation, the constraint being positive definiteness. It is the problem of completing a positive definite matrix for which only a well-ordered partial set of entries is given (and also giving necessary and sufficient conditions for the existence of the completion) or, alternatively, the problem of parametrizing positive definite matrices. This problem can be solved elegantly when the specified entries contain the main diagonal and further entries crowded along the main diagonal with a staircase boundary. This problem turns out to be equivalent to a constrained interpolation problem defined for a causal contractive matrix, with staircase entries again specified as before. The recursive solution calls for the development of a machinery known as scattering theory, which involves the introduction of nonpositive metrics and the use of *J*-unitary transformations where J is a sign matrix.

Chapter 14:

- **Chapter title:** The Scattering Picture

Keywords: scattering theory, theta matrices, indefinite metrics, bilinear representations, scattering matrices

Abstract: This chapter introduces and develops the scattering formalism, whose usefulness for interpolation has been demonstrated in Chapter 13, for the case of systems described by state-space realizations. This is in preparation for the next three chapters that use it to solve various further interpolation and embedding problems.

Chapter 15:

- **Chapter title:** Constrained Interpolation

Keywords: valuation, W-transform, interpolation, Nevanlinna–Pick, Schur, Hermite–Fejer

Abstract: The chapter shows how classical interpolation problems of various types (Schur, Nevanlinna–Pick, Hermite–Fejer) carry over and generalize to the time-variant and/or matrix situation. We show that they all reduce to a single generalized constrained interpolation problem, elegantly solved by time-variant scattering theory. An essential ingredient is the definition of the notion of valuation for time-variant systems, thereby generalizing the notion of valuation in the complex plane provided by the classical z-transform.

Chapter 16:

- **Chapter title:** Constrained Model Reduction

Keywords: model reduction, matrix Schur Takagi, AAK theory, Nehari completion

Abstract: This chapter provides for a further extension of constrained interpolation that is capable of solving the constrained model reduction problem, namely the generalization of Schur–Takagi-type interpolation to the time-variant setting. This remarkable result demonstrates the full power of time-variant system theory as developed in this book.

Chapter 17:

- **Chapter title:** Isometric Embedding for Causal Contractions

Keywords: inner embedding, contractive, canonical, Darlington synthesis

Abstract: The final chapter completes the scattering theory with an elementary approach to inner embedding of a contractive, quasi-separable causal system (in engineering terms: the embedding of a lossy or passive system in a lossless system, often called Darlington synthesis). Such an embedding is always possible in the finitely indexed case but does not generalize to infinitely indexed matrices. (This last issue requires more advanced mathematical methods and lies beyond the subject matter of the book.)

Chapter 18:

- **Chapter title:** Appendix: Data Model and Implementations

Keywords: data flow, computer architecture, functional specification, parallel architecture, AST model

Abstract: The appendix defines the data model used throughout the book and describes what can best be called an algorithmic design specification, that is, the functional and graphical characterization of an algorithm, chosen so that it can be translated to a computer architecture (be it in soft- or in hardware). We follow hereby a powerful "data flow model" that generalizes the classical signal flow graphs and that can be further formalized to generate the information necessary for the subsequent computer system design at the architectural level (i.e., the assignment of operations, data transfer and memory usage). The model provides for a natural link between mathematical operations and architectural representations. It is, at the same token, well adapted to the generation of parallel processing architectures.